AMBARZUMYAN TYPE THEOREM FOR A MATRIX QUADRATIC STURM-LIOUVILLE EQUATION WITH ENERGY-DEPENDENT POTENTIA

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Abstract : Ambarzumyan's theorem for quadratic Sturm-Liouville problem is extended to second order differen tems of dimension $d \geq 2$. It is shown that if the spectrum is the same as the spectrum belonging to the zero potent the matrix valued functions $P(x)$ and $Q(x)$ are both zero.

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1. Introduction

From an historical viewpoint, Ambarzumyan's paper may be thought of as the first paper in the th inverse spectral problems associated with Sturm-Liouville operators. Ambarzumyan proved the fo theorem:

If $\{n^2 : n = 0, 1, 2, ...\}$ is the spectral set of the boundary value problem

$$y'' + (\lambda - q(x))y = 0, \; y'(0) = y'(\pi) = 0$$

then $q = 0$ in $[0, \pi]$, where $q(x) \in L^2[0, \pi]$ [1]. Ambarzumyan's theorem was extended to the second differential systems of two dimensions in [2], to Sturm-Liouville differential systems of any dimen [3], to the Sturm–Liouville equation (which is concerned only with Neumann boundary conditions general boundary conditions by imposing an additional condition on the potential function [4], and multi-dimensional Dirac operator in [5]. In addition, some different results of Ambarzumyan's theore been obtained in [6], [7], [8], [9], [10].

Ambarzumyan's theorem was extended to the following boundary value problem by imposing to dition for p

$$-y'' + [2\lambda p(x) + q(x)]\, y = \lambda^2 y, x \in [0, \pi]$$

with the homogeneous Neumann boundary conditions

$$y'(0) = y'(\pi) = 0,$$

where λ is a spectral parameter and $p \in W_2^2[0,\pi]$ and $q \in W_2^1[0,\pi]$ by Koyunbakan, Lesnic and Pa
[11]. This problem is called the quadratic pencil of the Schrödinger operator. If $p(x) = 0$ the classical
Liouville operator is obtained. Some versions of the eigenvalue problem (1.2)-(1.3) were studied exte
in [12], [13], [14], [15], [16].

Before giving the main results, we mention some physical properties of the quadratic equatio
problem of describing the interactions between colliding particles is of fundamental interest in physic
interesting in collisions of two spinles particles, and it is supposed that the $s-$wave scattering matr
the $s-$wave binding energies are exactly known from collision experiments. For a radial static po
$V(E,x)$ and $s-$ wave, the Schrödinger equation is written as

$$y'' + [E - V(E,x)]\, y = 0,$$

where

$$V(E,x) = 2\sqrt{E}p(x) + q(x).$$

We note that with the additional condition $q(x) = -p^2(x)$, the above equation reduces to the Klein-
s-wave equation for a particle of zero mass and energy $\sqrt{E}$ [17].

2. Matrix Differential Equations

For simplicity, A_{ij} denotes entry of matrix A at the $i-th$ row and $j-th$ column and I_d is
identity matrix and 0_d is a $d \times d$ zero matrix.

We are interested in the eigenvalue problem

$$-\phi'' + [2\lambda P(x) + Q(x)]\, \phi = \lambda^2 \phi$$

$$A\phi(0) + B\phi'(0) = C\phi(\pi) + D\phi'(\pi) = 0,$$

where $P(x) = diag[p_1(x), p_2(x), ..., p_d(x)]$ and $Q(x)$ are $d \times d$ real symmetric matrix-valued functio
those $n \times n$ matrices A, B, C and D satisfy the following conditions

$$DC^* : \text{Self-Adjoint}$$

$$BA^* = 0$$

$$rank[A,B] = rank[C,D] = n.$$

To study (2.1)-(2.2), we introduce the following matrix differential equation

$$-Y'' + [2\lambda P(x) + Q(x)]\, Y = \lambda^2 Y, x \in [0,\pi]$$

$$Y(0,\lambda) = I_d, Y'(0,\lambda) = 0_d$$

where λ is a spectral parameter, $Y(x) = [y_k(x)], k = \overline{1,d}$ is a column vector, $P(x) = diag[p_1(x), p_2(x), .$
and $Q(x)$ are $d \times d$ real symmetric matrix-valued functions, $P \in W_2^2[0,\pi]$ and $Q \in W_2^1[0,\pi]$, where W
$(k = 1,2)$ denotes a set whose element is a $k-$th order continuously differentiable function in $L_2[0,\pi]$,
and $\mu = \sigma + it \in \mathbb{C}$. Then, λ is an eigenvalue of (2.1)-(2.2), if the matrix which is called charach
function

$$W(\mu) = CY(\pi,\mu) + DY'(\pi,\mu)$$

is singular.

In order to describe $W(\mu)$ explicitly, we must know how to express the solution $Y(x,\mu)$. The s
$Y(x,\mu)$ of (2.6)-(2.7) can be expressed as

$$Y(x,\mu) = \cos[\lambda I_d x - \alpha(x)] + \int_0^x A(x,t)\cos(\lambda t)dt + \int_0^x B(x,t)\sin(\lambda t)dt$$

where $A(x,t)$ and $B(x,t)$ are symmetric matrix-valued functions whose entries have continuous
derivatives up to order two respect to t and x, and it can described by the following lemma.

Lemma 2.1. [18] *Let A and B be as in (2.8). Then,* A and B satisfy following conditions

$$\begin{aligned}
\frac{\partial^2 A(x,t)}{\partial x^2} - 2P(x)\frac{\partial B(x,t)}{\partial t} - Q(x)A(x,t) &= \frac{\partial^2 A(x,t)}{\partial t^2} \\
\frac{\partial^2 B(x,t)}{\partial x^2} + 2P(x)\frac{\partial A(x,t)}{\partial t} - Q(x)B(x,t) &= \frac{\partial^2 B(x,t)}{\partial t^2}
\end{aligned}$$

$$A(0,0) = h_d, \quad B(x,0) = 0_d, \quad \left.\frac{\partial A(x,t)}{\partial t}\right|_{t=0} = 0_d,$$

with $\alpha(x) = \int_0^x P(t)dt$. Moreover, there holds

$$2[\cos\alpha(x)A(x,x) + \sin\alpha(x)B(x,x)] = 2h + \int_0^x T_1(t)dt$$

and

$$2[\sin\alpha(x)A(x,x) - \cos\alpha(x)B(x,x)] = P(x) - P(0) + \int_0^x T_2(t)dt$$

where

$$T_1(x) = P^2(x) + \cos\alpha(x)Q(x)\cos\alpha(x) + \sin\alpha(x)Q(x)\sin\alpha(x)$$

and

$$T_2(x) = \sin\alpha(x)Q(x)\cos\alpha(x) - \cos\alpha(x)Q(x)\sin\alpha(x).$$

Lemma 2.2. [18] *The eigenvalues of the operator* $L(P,Q;h,H)$ *are*

$$\lambda_n = n + \frac{\alpha_j}{\pi}, n = 0, \pm 1, \pm 2, \pm 3, ..., j = \overline{1,d} \text{ and } \alpha_j = \int_0^\pi p_j(x)dx.$$

3. Main Results

In this section, some uniqueness theorems are given for the equation (2.6) with the Robin bo
conditions. It is shown that an explicit formula of eigenvalues can determine the functions $Q(x)$ an
be zero both. Results are some generalizations of [11].

Consider the a second matrix quadratic initial value problem

$$-Y'' + \left[2\lambda P(x) + \widetilde{Q}(x)\right] Y = \lambda^2 Y, x \in [0,\pi]$$

$$Y(0,\lambda) = I_d, Y'(0,\lambda) = 0_d$$

where $\widetilde{Q}$ has the same properties of Q.

The problems (2.6)-(2.7) and (3.1)-(3.2) will be denoted by $L(P,Q;h,H)$ and $\widetilde{L}(P,\widetilde{Q};h,H)$ an
trums of these problems wil be denoted by $\sigma(P,Q)$ and $\widetilde{\sigma}(P,\widetilde{Q})$, respectively.

Theorem 3. 1. Suppose that $\sigma(P,Q) = \widetilde{\sigma}(P,\widetilde{Q})$ and $\alpha(\pi) = 0$, then $\int_0^\pi \left[Q(x) - \widetilde{Q}(x)\right] dx = 0$
everywhere on $[0,\pi]$.

Proof: Since $\sigma(P,Q) = \widetilde{\sigma}(P,\widetilde{Q})$, it follows that $\lambda_n \in \sigma(P,Q)$ are large eigenvalues. Then we ca
from (2.8) that

$$\begin{aligned} Y'(\pi,\lambda_n) &= -(\lambda_n I_d - P(\pi)) \sin\left[\lambda_n I_d \pi - \alpha(\pi)\right] + A(\pi,\pi)\cos(\lambda_n \pi) + B(\pi,\pi)\sin(\lambda_n\pi) \\ &\quad + \int_0^x A_x(\pi,t)\cos(\lambda_n t)\,dt + \int_0^x B_x(\pi,t)\sin(\lambda_n t)\,dt \end{aligned}$$

and similarly for the problem (3.1)-(3.2), we can write

$$\begin{aligned} \widetilde{Y}'(\pi,\lambda_n) &= -(\lambda_n I_d - P(\pi)) \sin\left[\lambda_n \pi - \alpha(\pi)\right] + \widetilde{A}(\pi,\pi)\cos(\lambda_n \pi) + \widetilde{B}(\pi,\pi)\sin(\lambda_n\pi) \\ &\quad + \int_0^x \widetilde{A}_x(\pi,t)\cos(\lambda_n t)\,dt + \int_0^x \widetilde{B}_x(\pi,t)\sin(\lambda_n t)\,dt. \end{aligned}$$

By substracting, $Y'(\pi,\lambda_n)$ and $\widetilde{Y}'(\pi,\lambda_n)$, we get

$$\begin{aligned} 0 &= \left[A(\pi,\pi) - \widetilde{A}(\pi,\pi)\right]\cos(\lambda_n\pi) + \left[B(\pi,\pi) - \widetilde{B}(\pi,\pi)\right]\sin(\lambda_n\pi) \\ &\quad + \int_0^x \left[A_x(\pi,t) - \widetilde{A}_x(\pi,t)\right]\cos(\lambda_n t)\,dt + \int_0^x \left[B_x(\pi,t) - \widetilde{B}_x(\pi,t)\right]\sin(\lambda_n t)\,dt. \end{aligned}$$

By using Riemann-Lebesque lemma and for $\lambda_n \to \infty$ in Lemma 2.2., we obtain $A(\pi,\pi) = \widetilde{A}(\pi,$
the other hand by Lemma 2.1., we know the following equalities,

$$2\frac{d}{dx}\left[\cos\alpha(x)A(x,x) + \sin\alpha(x)B(x,x)\right] = P^2(x) + Q(x)$$

$$2\frac{d}{dx}\left[\cos\alpha(x)\widetilde{A}(x,x) + \sin\alpha(x)\widetilde{B}(x,x)\right] = P^2(x) + \widetilde{Q}(x).$$

After substracting (3.3),(3.4) and integrating, we get

$$\int_0^\pi \left[Q(x) - \widetilde{Q}(x)\right] dx = 2\left\{\left[A(\pi,\pi) - \widetilde{A}(\pi,\pi)\right]\cos\alpha(\pi) + \left[B(\pi,\pi) - \widetilde{B}(\pi,\pi)\right]\sin\alpha(\pi)\right\}$$

and

$$\int_0^\pi \left[Q(x) - \widetilde{Q}(x)\right] dx = 0$$

where $\alpha(0) = \alpha(\pi) = 0$ and $A(\pi,\pi) = \widetilde{A}(\pi,\pi)$. This completes the proof.

Theorem 3. 2. Let $P(x) = diag[p_1(x), p_2(x), ..., p_d(x)]$ and $Q(x)$ are two $d \times d$ real symmetric
valued functions, and $\alpha(\pi) = 0$. If $\{0\} \cup \{m_j : j = 1, 2, ...\}$ is a subset of the spectrum of the fo
$d-$dimensional second order differential system

$$-\phi'' + [2\lambda P(x) + Q(x)]\,\phi = \lambda^2\phi,\ \ \phi'(0) = \phi'(\pi) = 0$$

where 0 is the first eigenvalue of (3.5), m_j is a strictly ascending infinite sequence of positive intege
0 and m_j are multiplicity of n, then $P(x) = Q(x) = 0$.

Proof: Suppose for the (3.5) Neumann problem, there are infinitely many eigenvalues of the fo
m_j are positive integers, $j = 1, 2, ...$ and each m_j is of multiplicity n. For such a case, Let $A = C =$
$B = D = I_d$ in (2.1). Then, we get

$$Y'(\pi, m_j) = 0.$$

On the other hand, by (2.8), we have

$$\begin{aligned} Y'(x,\lambda_n) &= -(\lambda_n I_d - P(x))\sin\left[\lambda_n x - \alpha(x)\right] + A(x,x)\cos(\lambda_n x) + B(x,x)\sin(\lambda_n x) \\ &\quad + \int_0^x A_x(x,t)\cos(\lambda_n t)\,dt + \int_0^x B_x(x,t)\sin(\lambda_n t)\,dt. \end{aligned}$$

Equations (3.6) and (3.7) imply

$$A(\pi,\pi)\cos(m_j\pi) + \int_0^\pi A_x(\pi,t)\cos(\lambda_n t)\,dt + \int_0^\pi B_x(\pi,t)\sin(\lambda_n t)\,dt = 0.$$

We have from (3.8) and Riemann Lebesque lemma that $A(\pi,\pi)=0$. Then, by integration of

$$Q(x)+P^2(x)=2\frac{d}{dx}\left[\cos\alpha(x)A(x,x)+\sin\alpha(x)B(x,x)\right],$$

we get

$$\cos\alpha(x)A(x,x)+\sin\alpha(x)B(x,x)=\frac{1}{2}\int\limits_0^x\left[Q(t)+P^2(t)\right]dt.$$

Since $\alpha(\pi)=0$, we have

$$0=A(\pi,\pi)=\frac{1}{2}\int\limits_0^\pi\left[Q(t)+P^2(t)\right]dt$$

and

$$\int\limits_0^\pi Q(x)dx=-\int\limits_0^\pi P^2(x)dx.$$

By using the reality of 0 being the ground state of the eigenvalue problem (3.5), we may find $n-$ l
independent constant vectors corresponding to the same eigenvalue 0 by the variational principle and
them by φ_j, $j=1,2,...,d$. Since they should the following equation

$$-\varphi_j''+\left[2\lambda_0P(x)+Q(x)\right]\varphi_j=\lambda_0^2\varphi_j.$$

Then, we obtain

$$Q(x)\varphi_j=0,\ 0\leq x\leq\pi.$$

Thus $Q(x)=0$. If we consider (3.9) and diagonally of P, we get $P(x)=0$. This completes the pr

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